The Relationship between Howusu and Schwarzschild Metric Tensors in Deriving Ricci Curvature Tensor for All Gravitational Fields in Nature

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ABSTRACT

The Howusu metric tensor which was said to describe the gravitational field for all gravitational fields in nature was used to derive the Ricci Curvature Tensor $R_{\mu\nu}$. Results obtained were compared with the Ricci Curvature Tensor $R_{\mu\nu}$ derived from the well-known Schwarzschild metric tensor. It was found that, at $r \to 0$, the Ricci Curvature Tensor for both metric tensors were different but behaved the same, as $r \to \infty$,

KEYWORDS: *Ricci Curvature Tensor, Howusu Metric Tensor, Schwarzschild Metric Tensor.*

INTRODUCTION

The Schwarzschild metric represents the gravitational field around a symmetrically spherical object without angular momentum. This Schwarzschild metric can be summarized as (Kumar,2009).

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{2M}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix} \quad (1)$$

where M is the mass of the object and r is the distance away from the object. In 2012, Howusu Metric Tensor was introduced and was said to be a solution to Einstein's field equations that describes the gravitational field for all gravitational field in nature. The metric is written as follows;

$$g_{\mu\nu} = \begin{pmatrix} -exp\left(\frac{2GM}{c^2r}\right) & 0 & 0 & 0 \\ 0 & exp\left(-\frac{2GM}{c^2r}\right) & 0 & 0 \\ 0 & 0 & r^2exp\left(-\frac{2GM}{c^2r}\right) & 0 \\ 0 & 0 & 0 & r^2sin^2\theta exp\left(-\frac{2GM}{c^2r}\right) \end{pmatrix} \quad (2)$$

where C is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object (Howusu, 2012). In this paper, the Ricci Curvature Tensor in the Spherical coordinate based upon the Howusu Metric Tensor will be derived, and then the results will be compared with the well-known Ricci Curvature Tensor in the Spherical Coordinate based on Schwarzschild Metric Tensor.

THEORY

$R_{\mu\nu}$ is the Ricci Curvature tensor, which is a contraction of the Riemann Curvature tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \quad (3)$$

where $R_{\mu\nu}$ is expressed explicitly as (Abalaka & Ekpe, 2021)

$$R_{\mu\nu} = R^{0}_{\mu 0\nu} + R^{1}_{\mu 1\nu} + R^{2}_{\mu 2\nu} + R^{3}_{\mu 3\nu} \quad (4)$$

and $R^{\alpha}_{\mu\alpha\nu}$ is the Riemman Curvature Tensor given by (Obaje & Ekpe,2021)

$$R^{\alpha}_{\mu\alpha\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha} \quad (5)$$

where $\Gamma^{\propto}_{\mu\nu}$ is the Christoffel symbol of the second kind pseudo tensor.(Obaje & Ekpe, 2020)

It should be noted that non-zero terms from the mathematical results of the calculation of Riemann Tensor only half the amount of Riemann Tensor will be listed based on the listed property:

$$R^{\alpha}_{\mu\alpha\nu} = -R^{\alpha}_{\mu\nu\alpha} \quad (6)$$

We have;

$$R^{0}_{101} = -\frac{2GM}{c^2r^3} \quad (7)$$

$$R^{0}_{202} = \frac{GM}{c^2r} - \frac{G^2M^2}{c^4r^2} \quad (8)$$

$$R^{0}_{303} = \frac{GM}{c^2r}sin^2\theta - \frac{G^2M^2}{c^4r^3}sin^2\theta \quad (9)$$

$$R^1_{010} = \frac{2GM}{c^2 r^3} \quad (10)$$

$$R^1_{212} = 1 - \frac{GM}{c^2 r} \quad (11)$$

$$R^1_{313} = -\frac{GM \sin^2\theta}{c^2 r} \quad (12)$$

$$R^2_{020} = \frac{GM}{c^4 r^3} - \frac{G^2 M^2}{c^4 r^4} \quad (13)$$

$$R^2_{121} = -\frac{GM}{c^2 r^3} \quad (14)$$

$$R^2_{323} = \frac{2GM \sin^2\theta}{c^2 r} - \frac{G^2 M^2 \sin^2\theta}{c^4 r^2} - \cos^2\theta \quad (15)$$

$$R^3_{030} = \frac{GM}{c^4 r^3} - \frac{G^2 M^2}{c^4 r^4} \quad (16)$$

$$R^3_{131} = -\frac{GM}{c^2 r^3} \quad (17)$$

$$R^3_{232} = -\frac{G^2 M^2}{c^4 r^2} \quad (18)$$

where c is the speed of light; G is the universal constant of gravitation; M is the mass of the object and r is the distance away from the object (Obaje & Ekpe, 2021). Having derived the Riemann tensor, one can now derive the Ricci tensor by substituting (7)-(18) into (4), which resulted into the following:

$$R_{00} = \frac{2GM}{c^2 r^3} - \frac{2G^2 M^2}{c^4 r^4} + \frac{2GM}{c^4 r^3} \quad (19)$$

$$R_{11} = -\frac{4GM}{c^2 r^3} \quad (20)$$

$$R_{22} = 1 - \frac{2G^2 M^2}{c^4 r^2} \quad (21)$$

$$R_{33} = \frac{2GM \sin^2\theta}{c^2 r} - \frac{G^2 M^2 \sin^2\theta}{c^4 r^3} - \frac{G^2 M^2 \sin^2\theta}{c^4 r^2} - \cos^2\theta \quad (22)$$

RESULTS AND DISCUSSION

The mathematical results from the calculation of the Ricci curvature tensor using the Howusu Metric Tensor are:

$$R_{00} = \frac{2GM}{c^2 r^3} - \frac{2G^2 M^2}{c^4 r^4} + \frac{2GM}{c^4 r^3} \quad (23)$$

$$R_{11} = -\frac{4GM}{c^2r^3} \tag{24}$$

$$R_{22} = 1 - \frac{2G^2M^2}{c^4r^2} \tag{25}$$

$$R_{33} = \frac{2GM\sin^2\theta}{c^2r} - \frac{G^2M^2\sin^2\theta}{c^4r^3} - \frac{G^2M^2\sin^2\theta}{c^4r^2} - \cos^2\theta \tag{26}$$

Comparing the Ricci curvature tensor in (23) to (26) with the well-known Schwarzschild's Ricci curvature tensor, it will be seen that the Ricci curvature tensor R_{00} of the Howusu Metric Tensor from equation (23) tends to infinity as $r \to 0$, and as $r \to \infty$, R_{00} tends to zero.

In the case of the Schwarzschild Metric Tensor, the Ricci curvature tensor R_{00} is:

$$R_{00} = 0 \tag{27}$$

Equating the Ricci curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor gives

$$\frac{2GM}{c^2r^3} - \frac{2G^2M^2}{c^4r^4} + \frac{2GM}{c^4r^3} = 0 \tag{28}$$

This is the condition for the Ricci curvature tensors R_{00} to be equal in the Howusu Metric Tensor and the Schwarzschild Metric Tensor.

The Ricci curvature tensor R_{11} of the Howusu Metric Tensor is

$$R_{11} = -\frac{4GM}{c^2r^3} \tag{29}$$

It is clear from equation (29) that, as $r \to 0$, $R_{11} = -\infty$ and as $r \to \infty$, $R_{11} \to 0$. For the Schwarzschild Metric Tensor Ricci curvature tensor R_{11} is:

$$R_{11} = 0 \tag{30}$$

Equating the Ricci curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor,

$$-\frac{4GM}{c^2r^3} = 0 \tag{31}$$

This condition can only be satisfied if $M = 0$ or $r = \infty$.

Considering the Ricci curvature tensor R_{22} of the Howusu Metric Tensor

$$R_{22} = 1 - \frac{2G^2M^2}{c^4r^2} \tag{32}$$

One finds out from (32) that, as $r \to 0$, $R_{22} = -\infty$ and as $r \to \infty$, $R_{22} \to 0$.

For the Schwarzschild Metric Tensor, the Ricci curvature tensor R_{22} is

$$R_{22} = 0 \tag{33}$$

By equating the Ricci curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor,

$$1-\frac{2G^2M^2}{c^4r^2}=0 \quad (34)$$

This condition can only be satisfied if $r=\frac{GM\sqrt{2}}{C^2}$.

Calculations carried out herein have shown the Ricci curvature tensor R_{33} of the Howusu Metric to be

$$R_{33}=\frac{2GM\sin^2\theta}{c^2r}-\frac{G^2M^2\sin^2\theta}{c^4r^3}-\frac{G^2M^2\sin^2\theta}{c^4r^2}-\cos^2\theta \quad (35)$$

From (35) it is evident that, as $r\to 0$, $R_{33}=-\infty$ and as $r\to\infty$, $R_{33}\to 0$.

For the Schwarzschild Metric Tensor, the Ricci curvature tensor R_{33} is

$$R_{33}=0 \quad (36)$$

Equating the Ricci curvature tensor of the Howusu Metric Tensor with the Schwarzschild Metric Tensor,

$$\frac{2GM\sin^2\theta}{c^2r}-\frac{G^2M^2\sin^2\theta}{c^4r^3}-\frac{G^2M^2\sin^2\theta}{c^4r^2}-\cos^2\theta=0 \quad (37)$$

This is the condition for the Ricci Curvature Tensor R_{33} to be equal in the two metric tensors.

CONCLUSION

In this paper, we formulated the Ricci Curvature Tensor based on Howusu Metric Tensor for all gravitational fields in nature (19) to (22). We compared Howusu's Ricci Curvature Tensor with the well-known Schwarzschild's Ricci Curvature Tensor and discovered that even though they acted alike at $r=\infty$ they differ at $r=0$. Although Howusu's Ricci Curvature Tensor is slightly different from Schwarzschild's Ricci Curvature Tensor, it is more generalized because there is no restriction on it unlike Schwarzschild making it cover more areas. The doors are henceforth open to more verification.

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